

On the Algebraic Montgomery-Yang Problem

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Main Topic

“How many singularities can the topologically simplest algebraic surfaces have?”

Algebraic Montgomery–Yang Problem (Kollár, '08)

Every $\mathbb{Q}$ -homology projective plane with quotient singularities and a simply-connected smooth locus has at most three singular points.

In this talk, we will explain how this conjecture is completely resolved by combining **algebraic geometry** and **smooth 4-manifold topology**.

Outline:

- 1 The algebraic Montgomery–Yang problem and its resolution
- 2 Topological background for the proof
- 3 Main ideas of the proof
- 4 Further questions

Part I

The Algebraic Montgomery–Yang problem and Its Resolution

Rational Homology Projective Planes

A normal projective complex surface S is called a **rational homology projective plane** (or $\mathbb{Q}$ -homology $\mathbb{CP}^2$) if

$$H_*(S; \mathbb{Q}) \cong H_*(\mathbb{CP}^2; \mathbb{Q}).$$

Equivalently,

$$(b_0(S), b_1(S), b_2(S), b_3(S), b_4(S)) = (1, 0, 1, 0, 1)$$

Thus, a $\mathbb{Q}$ -homology $\mathbb{CP}^2$ is a projective surface with the **simplest possible rational homology**.

Assume further that S has only quotient singularities and that $S^0 := S \setminus \text{Sing}(S)$ is simply-connected.

Question

Which collections of quotient singularities can occur on S ?

Question

In particular, how many quotient singularities can S have?

Algebraic Montgomery-Yang Problem

Algebraic Montgomery-Yang Problem (Kollár, 08)

Let S be a $\mathbb{Q}$ -homology $\mathbb{CP}^2$ with quotient singularities. If $S^0 := S \setminus \text{Sing}(S)$ is simply-connected, then S has at most 3 singular points.

Example. For pairwise relatively prime integers $a, b, c > 1$, the **weighted projective plane**

$$\mathbb{CP}(a, b, c) = (\mathbb{C}^3 \setminus \{0\})/\mathbb{C}^*, \quad \lambda \cdot (x, y, z) = (\lambda^a x, \lambda^b y, \lambda^c z),$$

is a $\mathbb{Q}$ -homology $\mathbb{CP}^2$ with simply-connected smooth locus, which has exactly three cyclic quotient singularities, located at the coordinate points.

Montgomery–Yang Problem

“How many non-free orbits can a pseudo-free S^1 -action on S^n have?”

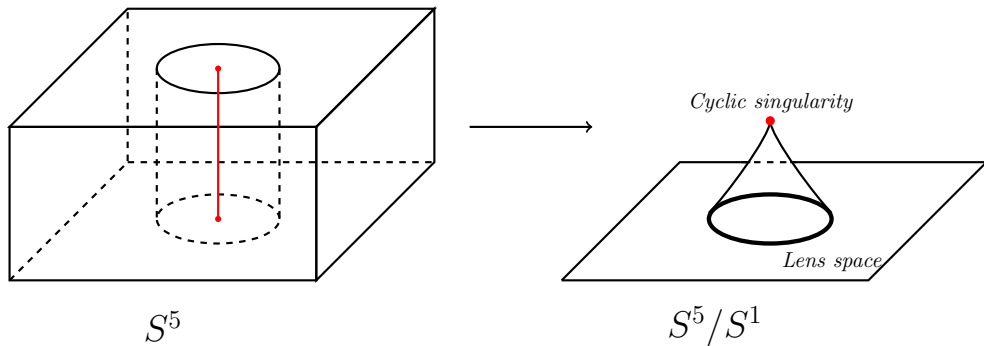
- **pseudo-free**: free except for finitely many non-free orbits
- For S^{2k} , the Euler characteristic forces fixed points; in the pseudo-free case there are exactly two fixed points.
- (Seifert, '32) A pseudo-free S^1 -action on S^3 has at most two non-free orbits.
- (Montgomery–Yang, '72) For any finite collection of cyclic groups $\mathbb{Z}_{p_1}, \dots, \mathbb{Z}_{p_m}$, there exists a pseudo-free S^1 -action on a homotopy 7-sphere whose non-free orbits have precisely these isotropy groups.
- (Petrie, '74) Similar phenomena occur in higher odd dimensions.

Montgomery–Yang Problem (Fintushel–Stern, '87)

Let $S^1 \times S^5 \rightarrow S^5$ be a pseudo-free smooth action. Then it has at most 3 non-free orbits.

From the Montgomery–Yang Problem to Algebraic Geometry

The Montgomery–Yang problem concerns pseudo-free S^1 -actions on S^5 . In the orbit space, non-free orbits give rise to cyclic quotient singularities, whose links are lens spaces.



This picture inspired Kollár's algebraic analogue for $\mathbb{Q}$ -homology projective planes with **quotient singularities**.

Known Results toward the Algebraic Montgomery–Yang Problem

Let S be a $\mathbb{Q}$ -homology $\mathbb{CP}^2$ with quotient singularities and $\pi_1(S^0) = 1$.

- By the orbifold BMY inequality,

$$|\mathrm{Sing}(S)| \leq 4.$$

- Since S is a $\mathbb{Q}$ -homology $\mathbb{CP}^2$, its Picard number is $\rho(S) = 1$. Thus one is led to the following three cases:

$$-K_S \text{ ample}, \quad K_S \equiv 0, \quad K_S \text{ ample}.$$

These correspond respectively to the log del Pezzo, log Enriques, and log general type cases.

- (Hwang-Keum, '11, '13, '14) It is proved that

$$|\mathrm{Sing}(S)| \leq 3$$

in the following cases:

- ▶ S has at least one non-cyclic quotient singularity;
- ▶ S is not rational, which includes the numerically trivial case;
- ▶ $-K_S$ is ample.

Therefore, the remaining case is when S is rational, has only **four cyclic quotient singularities**, and K_S is **ample**.

Possible Orders from the Orbifold BMY Inequality

Let S be a $\mathbb{Q}$ -homology $\mathbb{CP}^2$ with four cyclic quotient singularities and $\pi_1(S^0) = 1$, and let $p_i = |G_i|$ be the orders of the local fundamental groups at the singular points, with $p_1 \leq p_2 \leq p_3 \leq p_4$.

Then S satisfies the **orbifold BMY inequality** (Bogomolov–Miyaoka–Yau inequality):

$$0 \leq e_{\text{orb}}(S) = e(S) - \sum_{p \in \text{Sing}(S)} \left(1 - \frac{1}{|G_p|}\right) = -1 + \sum_{p \in \text{Sing}(S)} \frac{1}{p_i}.$$

If K_S is ample, then

$$0 < K_S^2 \leq 3e_{\text{orb}}(S).$$

Together with the fact that the p_i 's are pairwise relatively prime, this implies that (p_1, p_2, p_3, p_4) is one of the following:

- ① $(2, 3, 5, p_4)$, where $p_4 \geq 7$ and $\gcd(p_4, 30) = 1$;
- ② $(2, 3, 7, p_4)$, where $p_4 \in \{11, 13, 17, 19, 23, 25, 29, 31, 37, 41\}$;
- ③ $(2, 3, 11, 13)$.

Main Results

We say (p_1, p_2, p_3, p_4) or $\left\{ \frac{1}{p_1}(1, q_1), \frac{1}{p_2}(1, q_2), \frac{1}{p_3}(1, q_3), \frac{1}{p_4}(1, q_4) \right\}$ **is realized by** a $\mathbb{Q}$ -homology $\mathbb{CP}^2$ if $\exists$ a $\mathbb{Q}$ -homology $\mathbb{CP}^2$ with four cyclic singularities of the corresponding types.

Theorem (Jo–Park–P., '24)

If (p_1, p_2, p_3, p_4) is realized by a $\mathbb{Q}$ -homology $\mathbb{CP}^2$ with $\pi_1(S^0) = 1$, then

$$(p_1, p_2, p_3, p_4) = (2, 3, 5, p_4)$$

with $p_4 \geq 2599$.

We used tools from smooth 4-manifold topology, such as Donaldson's diagonalization theorem, linking forms, and Heegaard Floer d -invariants.

Main Results

In our recent work, we refine the Donaldson obstruction by imposing compatibility with the distinguished spin^c structure determined by the canonical class.

Theorem (Jo–Park–P., '26)

No $\mathbb{Q}$ -homology $\mathbb{CP}^2$ with $\pi_1(S^0) = 1$ can have four cyclic quotient singularities whose local fundamental group orders are

$$(2, 3, 5, p_4).$$

Combining this with the previous reductions, we obtain:

Theorem (Algebraic Montgomery–Yang Problem)

Let S be a $\mathbb{Q}$ -homology $\mathbb{CP}^2$ with quotient singularities. If $\pi_1(S^0) = 1$, then S has at most three singular points.

Main Results

Our argument only uses the weaker assumption

$$H_1(S^0; \mathbb{Z}) = 0.$$

Thus, in the cyclic case, we obtain the following slightly stronger statement.

Theorem (Jo–Park–P., '26)

Let S be a $\mathbb{Q}$ -homology $\mathbb{CP}^2$ with only cyclic quotient singularities. If $H_1(S^0; \mathbb{Z}) = 0$, then S has at most three singular points.

The restriction to cyclic quotient singularities is necessary: the statement fails if non-cyclic singularities are allowed (Hwang–Keum, 2011).

Part II

Topological Background for the Proof

Overview of the Proof

Topological Part.

- Starting from S , we construct a **closed negative definite 4-manifold** W_S .
- **Donaldson's diagonalization theorem**, together with the **distinguished spin^c structure** determined by the canonical class of S , gives a strong lattice-theoretic constraint on the possible singularity types.

Combinatorial part.

- For four cyclic quotient singularities of types

$$\frac{1}{2}(1, 1), \quad \frac{1}{3}(1, 1), \quad \frac{1}{5}(1, q_3), \quad \frac{1}{p_4}(1, q_4),$$

with $q_3 \in \{1, 3, 4\}$, $\gcd(30, p_4) = 1$, and $\gcd(p_4, q_4) = 1$, we combine this **lattice-theoretic constraint** with the **orbifold BMY inequality** and carry out a case-by-case analysis to rule out all remaining four-singularity cases.

Lattices and Lattice Embeddings

Definition

A **lattice** (L, Q) is a finitely generated free abelian group $L \cong \mathbb{Z}^n$ equipped with a symmetric bilinear form

$$Q: L \times L \rightarrow \mathbb{Z}.$$

- Q is **positive** (resp. negative) **definite** if $Q(x, x) > 0$ (resp. $Q(x, x) < 0$) for all nonzero $x \in L$.
- Q is **unimodular** if $\det(Q) = \pm 1$.
- A homomorphism

$$\iota: (L_1, Q_1) \hookrightarrow (L_2, Q_2)$$

is a **lattice embedding** if it is injective and preserves the forms:

$$Q_1(x, y) = Q_2(\iota(x), \iota(y)).$$

Example: The **standard negative definite lattice** $-\mathbb{Z}^n$ is the lattice $(\mathbb{Z}^n, \langle \cdot, \cdot \rangle)$ given by

$$\langle e_i, e_j \rangle = -\delta_{i,j}.$$

Intersection Form

Definition

Let X be a compact, oriented 4-manifold, possibly with boundary. Its **intersection form** is the symmetric bilinear form

$$Q_X: H_2(X; \mathbb{Z}) / \text{Tor} \times H_2(X; \mathbb{Z}) / \text{Tor} \longrightarrow \mathbb{Z}$$

defined by

$$Q_X(a, b) = \langle PD(a) \cup PD(b), [X, \partial X] \rangle.$$

- We denote the lattice $(H_2(X; \mathbb{Z}) / \text{Tor}, Q_X)$ by Q_X .
- If X is closed, then Poincaré duality implies that Q_X is unimodular.

Donaldson's Diagonalization Theorem

(Freedman, '82) Given any unimodular symmetric bilinear form, there exists a simply-connected, closed, **topological** 4-manifold whose intersection form is isometric to it.

Theorem (Donaldson's diagonalization theorem, '82, '87)

If the intersection form Q_W of a closed, oriented, **smooth** 4-manifold W is **negative definite**, then

$$Q_W \cong -\mathbb{Z}^{b_2(W)},$$

i.e., Q_W is diagonalizable over $\mathbb{Z}$.

Isometry classes of unimodular definite lattices

	even ($\Rightarrow 8 \sigma$)							odd
	rk	8	16	24	32	40	...	
definite	#	1	2	24	$> 10^7$	$> 10^{51}$	...	$-\mathbb{Z}^n, -\mathbb{Z}^n \oplus E_8, \dots$

Example: The topological E_8 -manifold is not smoothable.

Cyclic Quotient Singularities

Let $p > 1$ and $1 \leq q < p$ with $\gcd(p, q) = 1$. Set $\zeta = e^{2\pi i/p}$ and define

$$g = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^q \end{pmatrix} \in GL(2, \mathbb{C}), \quad G = \langle g \rangle \cong \mathbb{Z}/p\mathbb{Z}.$$

Then G acts linearly on $\mathbb{C}^2$ by

$$g \cdot (x, y) = (\zeta x, \zeta^q y).$$

The quotient

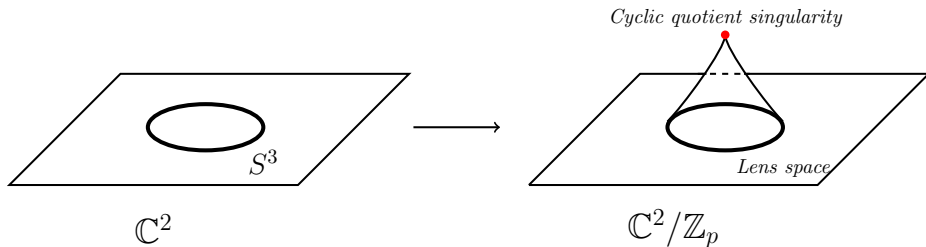
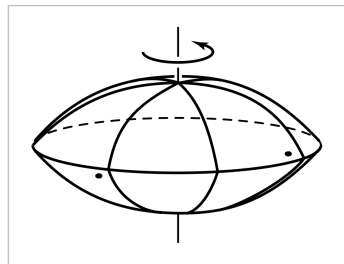
$$X = \mathbb{C}^2/G$$

has an isolated singularity at the image of the origin. This is called a **cyclic quotient singularity** of type

$$\frac{1}{p}(1, q).$$

Cyclic Quotient Singularities and Lens Spaces

The small neighborhood of the cyclic singularity of type $\frac{1}{p}(1, q)$ is homeomorphic to the cone on the **lens space** $L(p, q)$.



In other words, $L(p, q)$ is the link of a cyclic quotient singularity of type $\frac{1}{p}(1, q)$.

The canonical definite fillings of lens spaces

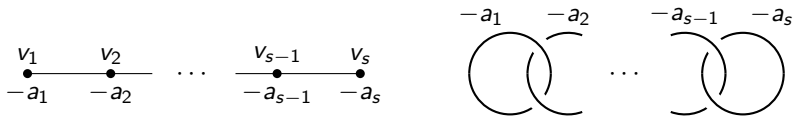
The canonical definite filling

Any lens space bounds a **negative definite** smooth 4-manifold.

- Expand p/q into its uniquely determined negative (Hirzebruch-Jung) continued fraction as follows:

$$\frac{p}{q} = [a_1, \dots, a_s] := a_1 - \frac{1}{a_2 - \frac{1}{\dots - \frac{1}{a_s}}} \quad (a_i \geq 2).$$

- $L(p, q)$ is the boundary of the **canonical plumbed 4-manifold** $X(p, q)$ constructed from the following linear graph:



The intersection form of $X(p, q)$ is represented by the incidence matrix of the linear graph, which is negative definite.



We say $\{v_1, \dots, v_s\}$ the **standard basis** of $Q_{X(p, q)}$ and v_s **right-end**.

Spin^c Structures

For $n \geq 3$, we have $\pi_1(SO(n)) \cong \mathbb{Z}/2$. Thus $\text{Spin}(n)$ is the universal double cover of $SO(n)$:

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow \text{Spin}(n) \longrightarrow SO(n) \longrightarrow 1.$$

A **spin structure** on an oriented Riemannian n -manifold is a lift of the $SO(n)$ -frame bundle through this double cover.

The group $\text{Spin}^c(n)$ is obtained by adding a $U(1)$ -factor:

$$\text{Spin}^c(n) = \frac{\text{Spin}(n) \times U(1)}{\{\pm(1, 1)\}}.$$

A **spin^c structure** is a lift of the $SO(n)$ -frame bundle to $\text{Spin}^c(n)$.

It has a determinant line bundle, whose first Chern class is denoted by $c_1(\mathfrak{s})$.

Spin^c Structures as H^2 -torsors

Let X be a smooth compact oriented 4-manifold. The set $\text{Spin}^c(X)$ is an affine space over $H^2(X; \mathbb{Z})$:

$$\text{Spin}^c(X) \times H^2(X; \mathbb{Z}) \longrightarrow \text{Spin}^c(X), \quad (\mathfrak{s}, \alpha) \mapsto \mathfrak{s} + \alpha.$$

Moreover,

$$c_1(\mathfrak{s} + \alpha) = c_1(\mathfrak{s}) + 2\alpha.$$

Thus, after choosing one spin^c structure as an origin, one can identify $\text{Spin}^c(X)$ with $H^2(X; \mathbb{Z})$ (not canonical).

The same holds for a boundary 3-manifold Y .

If $Y = \partial X$, the restriction map is compatible with $i^*: H^2(X; \mathbb{Z}) \rightarrow H^2(Y; \mathbb{Z})$:

$$(\mathfrak{s} + \alpha)|_Y = \mathfrak{s}|_Y + i^* \alpha.$$

Example. For a lens space $L(p, q)$,

$$\text{Spin}^c(L(p, q)) \cong \mathbb{Z}/p\mathbb{Z},$$

with the standard labeling due to Ozsváth–Szabó denoted by $\mathfrak{s}_{p,q}(i)$ for $i = 0, \dots, p-1$.

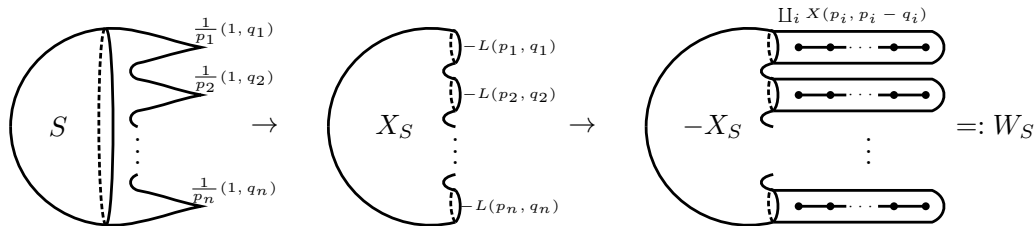
From Algebraic Surfaces to Smooth Definite 4-manifolds

Let S be a $\mathbb{Q}$ -homology $\mathbb{CP}^2$ with cyclic quotient singularities: $\frac{1}{p_1}(1, q_1), \dots, \frac{1}{p_n}(1, q_n)$.

Removing small cone neighborhoods of the singular points gives a smooth compact 4-manifold X_S with

$$b_2(X_S) = b_2^+(X_S) = 1, \quad \partial X_S \cong \coprod_{i=1}^n -L(p_i, q_i).$$

After reversing orientation and gluing the negative definite plumbings $X(p_i, p_i - q_i)$, we obtain a **closed negative definite smooth** 4-manifold $W_S := (-X_S) \cup_f \left(\coprod_{i=1}^n X(p_i, p_i - q_i) \right)$.



Gluing Map and spin^c Structures

When constructing W_S , we use a specific orientation-reversing diffeomorphism

$$f = \coprod_{i=1}^n f_{p_i, q_i} : \coprod_{i=1}^n L(p_i, q_i) \longrightarrow \coprod_{i=1}^n L(p_i, p_i - q_i),$$

where

$$f_{p,q}[z_1, z_2] = [\overline{z_1}, z_2].$$

Proposition

Under the Ozsváth–Szabó identification $\text{Spin}^c(L(p, q)) \cong \mathbb{Z}/p\mathbb{Z}$, the induced map on spin^c structures satisfies

$$f_{p,q}^*(\mathfrak{s}_{p,p-q}(i)) = \mathfrak{s}_{p,q}(p - 1 - i).$$

In particular,

$$f_{p,q}^*(\mathfrak{s}_{p,p-q}(0)) = \mathfrak{s}_{p,q}(p - 1),$$

which is precisely the boundary spin^c structure coming from the canonical class of S .

Intersection Form of W_S

By Donaldson's diagonalization theorem,

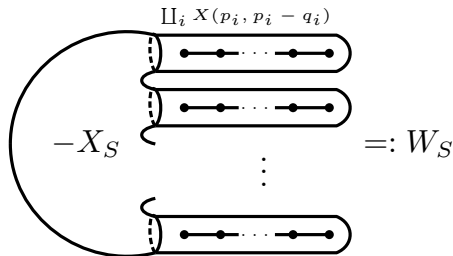
$$Q_{W_S} \cong -\mathbb{Z}^N,$$

where

$$N = \sum_i b_2(X(p_i, p_i - q_i)) + 1.$$

In particular, the inclusion of the plumbing pieces gives a lattice embedding

$$\iota : \bigoplus_{i=1}^n Q_{X(p_i, p_i - q_i)} \hookrightarrow -\mathbb{Z}^N.$$

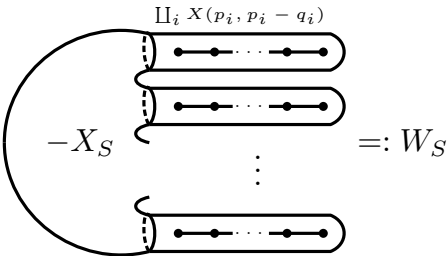


Intersection Form of W_S

Moreover, the canonical class of S determines a distinguished spin^c structure on X_S whose determinant line bundle is the canonical line bundle.

A detailed analysis of the boundary restrictions of spin^c structures and the pullback under the gluing map f shows that this structure extends over W_S . Let $\mathfrak{s}_K$ denote this extension.

The class $PD(c_1(\mathfrak{s}_K)) \in H_2(W_S; \mathbb{Z})$ then gives an additional constraint on the lattice embedding ι .



Lattice-Theoretic Constraint

Theorem (Jo–Park–P., '26)

Let S be a rational homology projective plane with $H_1(S^0; \mathbb{Z}) = 0$ and with n cyclic quotient singularities of types $\frac{1}{p_1}(1, q_1), \dots, \frac{1}{p_n}(1, q_n)$. Assume that, for each $i = 1, \dots, n$, either p_i is odd or $\sigma(q_i, p_i, +1) \neq \sigma(q_i, p_i, -1)$. Then there exists a codimension-one lattice embedding

$$\iota: \bigoplus_{i=1}^n Q_{X(p_i, p_i - q_i)} \longrightarrow -\mathbb{Z}^N = \langle -1 \rangle^N,$$

where $N = \sum_{i=1}^n b_2(X(p_i, p_i - q_i)) + 1$, satisfying the following conditions:

$$\sigma(q, p, \epsilon) = \frac{1}{p} \sum_{k=1}^{p-1} \left(\cot \frac{\pi k}{p} \cot \frac{\pi k q}{p} + 2\epsilon^k \csc \frac{\pi k}{p} \csc \frac{\pi k q}{p} \right) \quad (\epsilon \in \{\pm 1\}).$$

- For even p , $\sigma(q, p, +1) = \sigma(q, p, -1) \iff$ Two spin structures of the lens space $L(p, q)$ have the same Heegaard Floer d -invariant.
- In particular, $\sigma(1, 2, +1) \neq \sigma(1, 2, -1)$, so the theorem applies to the remaining $(2, 3, 5, p_4)$ cases.

Lattice-Theoretic Constraint (continued)

Theorem (continued)

- ① *There exists a vector $\mathbf{K} \in -\mathbb{Z}^N$ of the form*

$$\mathbf{K} = \epsilon_1 \mathbf{e}_1 + \cdots + \epsilon_{i_0-1} \mathbf{e}_{i_0-1} + 3\epsilon_{i_0} \mathbf{e}_{i_0} + \epsilon_{i_0+1} \mathbf{e}_{i_0+1} + \cdots + \epsilon_N \mathbf{e}_N \quad (1 \leq i_0 \leq N, \epsilon_i \in \{\pm 1\})$$

such that for any standard basis vector $v \in \bigoplus_{i=1}^n Q_{X(p_i, p_i - q_i)}$,

$$\mathbf{K} \cdot \iota(v) = \begin{cases} -v^2 & \text{if } v \text{ is right-end,} \\ -v^2 - 2 & \text{otherwise.} \end{cases}$$

- ② *The orthogonal complement of the image of the embedding ι is generated by a vector in $-\mathbb{Z}^N$ of square $-p_1 \cdots p_n$.*

The vector $\mathbf{K}$ is the image, under the Donaldson diagonalization $Q_{W_S} \cong -\mathbb{Z}^N$, of the Poincaré dual of $c_1(\mathfrak{s}_K)$ on W_S .

Example: $\mathbb{CP}(2, 3, 5)$

$\mathbb{CP}(2, 3, 5)$ has three cyclic quotient singularities $\frac{1}{2}(1, 1)$, $\frac{1}{3}(1, 1)$, and $\frac{1}{5}(1, 4)$, and $H_1(S^0; \mathbb{Z}) = 0$.

There exists a codimension-one lattice embedding

$$\iota: Q_{X(2,1)} \oplus Q_{X(3,2)} \oplus Q_{X(5,1)} \hookrightarrow -\mathbb{Z}^5$$

$$Q_X(2, 1) \oplus Q_X(3, 2) \oplus Q_X(5, 1)$$

with the standard basis $\{e_1, \dots, e_5\}$ of $-\mathbb{Z}^5$ such that

$$\begin{array}{c} v_1 \\ \bullet \\ -2 \end{array}$$

$$\iota(v_1) = e_1 - e_2,$$

$$\iota(v_2) = e_3 - e_4,$$

$$\iota(v_3) = e_4 - e_5,$$

$$\iota(v_4) = e_1 + e_2 + e_3 + e_4 + e_5,$$

$$\begin{array}{cc} v_2 & v_3 \\ \bullet & \bullet \\ -2 & -2 \end{array}$$

and

$$\mathbf{K} = -3e_1 - e_2 - e_3 - e_4 + e_5.$$

satisfying $\mathbf{K} \cdot \iota(v_i) = -v_i^2$ for $i = 1, 3, 4$ and $\mathbf{K} \cdot \iota(v_2) = -v_2^2 - 2$, and

$$\begin{array}{c} v_4 \\ \bullet \\ -5 \end{array}$$

$$(\text{Im } \iota)^\perp = \langle w \rangle, \text{ where } w = 3(e_1 + e_2) - 2(e_3 + e_4 + e_5) \text{ with } w \cdot w = -30.$$

Part III

Main Ideas of the Proof

Extending spin^c Structures over W_S

The canonical class of S determines a distinguished spin^c structure $\bar{\mathfrak{s}}_{\text{can}}$ on X_S . For a singularity of type $\frac{1}{p}(1, q)$, its boundary restriction is

$$\bar{\mathfrak{s}}_{\text{can}}|_{L(p,q)} = \mathfrak{s}_{p,q}(p-1),$$

where $\mathfrak{s}_{p,q}(i)$ denotes the Ozsváth–Szabó labeling of $\text{Spin}^c(L(p, q))$.

On the plumbing $X(p, p-q)$, we choose the spin^c structure $\mathfrak{s}_{K_{p,p-q}}$ whose evaluations on the standard basis vectors are

$$\langle c_1(\mathfrak{s}_{K_{p,p-q}}), v_i \rangle = \begin{cases} a_i - 2 = -v_i^2 - 2, & i < s, \\ a_s = -v_s^2, & i = s. \end{cases}$$

Its restriction to the boundary is

$$\mathfrak{s}_{p,p-q}(0) \in \text{Spin}^c(L(p, p-q)).$$

The gluing map satisfies

$$f_{p,q}^*(\mathfrak{s}_{p,p-q}(0)) = \mathfrak{s}_{p,q}(p-1).$$

Therefore the boundary spin^c structures match and glue to a structure $\mathfrak{s}_K \in \text{Spin}^c(W_S)$.

The form of $\mathbf{K}$

Let

$$\mathbf{K} := PD(c_1(\mathfrak{s}_K)) \in Q_{W_S} \cong -\mathbb{Z}^N = \langle e_1, \dots, e_N \rangle.$$

Write

$$\mathbf{K} = a_1 e_1 + \dots + a_N e_N.$$

Since $c_1(\mathfrak{s}_K)$ is characteristic, each coefficient a_i is odd.

A **square computation** gives

$$\mathbf{K}^2 = -N - 8.$$

Hence

$$\sum_{i=1}^N a_i^2 = N + 8.$$

Since all a_i 's are odd, this is possible only when exactly one coefficient is ± 3 and all the others are ± 1 .

Proof of the main theorem

Theorem (Jo-Park-P., '26)

There is no $\mathbb{Q}$ -homology $\mathbb{CP}^2$ with $H_1(S^0; \mathbb{Z}) = 0$ and ample canonical divisor having four cyclic quotient singularities of types

$$\frac{1}{2}(1, 1), \quad \frac{1}{3}(1, 1), \quad \frac{1}{5}(1, q_3), \quad \frac{1}{p_4}(1, q_4).$$

The proof is purely combinatorial: the orbifold BMY inequality and the lattice-theoretic constraints rule out the existence of such a lattice embedding.

$$\begin{array}{c}
 \bullet \\
 -2 \\
 \\
 \bullet \text{---} \bullet \\
 -2 \quad -2 \\
 \\
 \bullet \quad \text{or} \quad \bullet \text{---} \bullet \quad \text{or} \quad \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \\
 -5 \quad \quad -3 \quad -2 \quad \quad -2 \quad -2 \quad -2 \quad -2 \\
 \\
 \bullet \text{---} \bullet \quad \cdots \quad \bullet \text{---} \bullet \\
 -a_1 \quad -a_2 \quad \quad \quad -a_{s-1} \quad -a_s
 \end{array}
 \hookrightarrow -\mathbb{Z}^N$$

Role of the Strong Orbifold BMY Inequality

Since K_S is ample, the orbifold BMY inequality gives $0 < K_S^2 \leq 3e_{\text{orb}}(S)$.

In our case,

$$e_{\text{orb}}(S) = 3 - \left(1 - \frac{1}{2}\right) - \left(1 - \frac{1}{3}\right) - \left(1 - \frac{1}{5}\right) - \left(1 - \frac{1}{p_4}\right) = \frac{1}{30} + \frac{1}{p_4}.$$

Thus

$$0 < K_S^2 \leq \frac{1}{10} + \frac{3}{p_4}.$$

Write

$$\frac{p_4}{p_4 - q_4} = [a_1, \dots, a_s], \quad l_4 := \sum_{i=1}^s (a_i - 3).$$

The formula for K_S^2 and strong orbifold BMY inequality imply

$$l_4 = \begin{cases} 5 \text{ or } 6, & q_3 = 1, \\ 2 \text{ or } 3, & q_3 = 3, \\ 0 \text{ or } 1, & q_3 = 4. \end{cases}$$

This small list is what makes the final case-by-case lattice analysis manageable.

Part IV

Further Questions

Further Questions

Montgomery–Yang Problem (K3, Problem 5.7)

Does every pseudo-free smooth S^1 -action on S^5 have at most three non-free orbits?

Our method does not seem to apply directly to this topological problem: it uses genuinely algebro-geometric input, namely the orbifold BMY inequality and the canonical structure on S .

Three Fibers Conjecture (K3, Problem 4.58)

A Seifert fibered homology 3-sphere with more than three exceptional fibers cannot bound a homology 4-ball.

Classification Problem

Classify all possible collections of quotient singularities on $\mathbb{Q}$ -homology $\mathbb{CP}^2$ with simply-connected smooth locus.

Index of quotient singularities

Let S be a normal projective surface with quotient singularities.

Definition

(1) For a singularity $p \in S$, the **index** of p is the positive integer defined by

$$\text{index}(p) := \min\{k \in \mathbb{N} : kK_S \text{ is Cartier in a neighborhood of } p\}.$$

If locally $(S, p) \cong (\mathbb{C}^2/G, 0)$, then

$$\text{index}(p) = [G : G \cap \text{SL}(2, \mathbb{C})].$$

(2) The **index** of S is the positive integer defined by

$$\text{index}(S) := \min\{k \in \mathbb{N} : kK_S \text{ is Cartier}\} = \text{lcm}_{p \in \text{Sing}(S)} \text{index}(p).$$

Classification Results

The quotient singularities on $\mathbb{Q}$ -homology $\mathbb{CP}^2$'s of index ≤ 3 were classified by several authors, including Miyanishi–Zhang ('88), Alexeev–Nikulin('88-89), Kojima('03), and Zhang('89), under the assumption that $-K_S$ is ample.

We remove this ampleness assumption and give a complete classification of the singularity types of $\mathbb{Q}$ -homology $\mathbb{CP}^2$'s of index ≤ 2 with simply-connected smooth locus.

A $\mathbb{Q}$ -homology $\mathbb{CP}^2$ of index 1 is called **Gorenstein**.

Theorem (Jo–Park–P., '24)

Let S be a Gorenstein $\mathbb{Q}$ -homology $\mathbb{CP}^2$. If $\pi_1(S^0) = 1$, then the singularity type of S is one of the following seven types, and each of them is realizable:

$$E_8, \quad E_7, \quad E_6, \quad D_5, \quad A_4, \quad A_2A_1, \quad A_1.$$

Classification Results

Theorem (Jo-Park-P. '24)

Let S be a $\mathbb{Q}$ -homology $\mathbb{CP}^2$ of index 2. If $\pi_1(S^0) = 1$, then the singularity type of S is one of the following four types, and each of them is realizable:

$$K_5, \quad K_2A_2, \quad K_1A_4, \quad K_1.$$

Theorem (Jo-Park-P. '24)

Let S be a $\mathbb{Q}$ -homology $\mathbb{CP}^2$ of index 3. If $\pi_1(S^0) = 1$, then the singularity type of S is one of the following 18 types. All types, except possibly for $A_2(1,2)E_7$ and $A_2(2,2)E_8$, are realizable:

$$\begin{aligned} &A_1(1)E_8, \quad A_1(1)D_7, \quad A_1(1)A_6, \quad A_1(1)A_4A_1, \quad A_1(1)A_3, \quad A_1(1), \\ &A_6(1,1), \quad A_{10}(1,1), \quad A_2(1,2)E_7, \quad A_2(1,2)A_4, \quad A_2(1,2)A_1, \quad A_4(1,2)A_1, \\ &A_1(2)A_6, \quad A_1(2), \quad A_2(2,2)E_8, \quad A_2(2,2)A_3, \quad A_6(2,2), \quad D_5(2). \end{aligned}$$

The same result holds if we replace $\pi_1(S^0) = 1$ with $H_1(S^0; \mathbb{Z}) = 0$.

Remarks on the Index 3 Case

Remark

- ① For each of the 16 realizations, $-K_S$ is ample.
- ② Among these 16 types, the 10 types $A_1(1)A_4A_1$, $A_1(1)A_3$, $A_1(1)$, $A_2(1,2)A_4$, $A_2(1,2)A_1$, $A_4(1,2)A_1$, $A_1(2)A_6$, $A_1(2)$, $A_2(2,2)A_3$, and $D_5(2)$ cannot be realized by a $\mathbb{Q}H\mathbb{CP}^2$ with ample K_S .
- ③ Hwang (unpublished) has verified that the unknown 2 types $A_2(1,2)E_7$ and $A_2(2,2)E_8$ cannot be realized by a $\mathbb{Q}H\mathbb{CP}^2$ with anti-ample K_S .

Thank You!